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New perspectives on Yang-Mills theories with sixteen supersymmetries

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Abstract: We describe various approaches that give matrix descriptions of compactified NS five-branes. As a result, we obtain matrix models for Yang-Mills theories with sixteen supersymmetries in dimensions $2, 3, 4$ and 5. The equivalence of the various approaches relates the Coulomb branch of certain gauge theories to the moduli space of instantons on $\mathbf{T}^4$. We also obtain an equivalence between certain six-dimensional string theories. Further, we discuss how various perturbative and non-perturbative features of these Yang-Mills theories appear in their matrix formulations. The matrix model for four-dimensional Yang-Mills is manifestly S-dual. In this case, we describe how electric fluxes, magnetic fluxes and the interaction between vector particles are realized in the matrix model.

Keywords: Nonperturbative Approaches, Duality in Gauge Field Theories, M(atrix) Theories, Extended Supersymmetry.

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Contents

1 Introduction

The matrix model formulation of M theory [1], holds the promise of providing a non-perturbative description of M theory, and consequently string theory. Perhaps of equal importance, this description of M theory suggests that quantum field theory might be formulated in a new way. In this new description, some symmetries, such as Lorentz invariance, may no longer be manifest; yet other properties like S-duality may become manifest. The aim of this paper is to propose non-perturbative matrix formulations of Yang-Mills theories in various dimensions quantized in the discrete light-cone formalism (DLCQ) [2]. We will consider theories with sixteen supersymmetries,[1] although the main features of our construction seem to go over to cases with less supersymmetry. The case we understand best is Yang-Mills at a fixed point, and our discussion will focus on the matrix formulations of these conformal field theories.

Let us start with a configuration of parallel M theory five-branes. Recall that the theory on k coincident M theory five-branes is the $(2,0)$ field theory, first found in [3, 4]. At low energies, this theory flows to an interacting superconformal fixed point in six dimensions. See [5] for a review. After wrapping the five-branes on a circle, we obtain a theory which at low energies is five-dimensional Yang-Mills with a $U(k)$ gauge group and minimal matter content. The square of the Yang-Mills coupling constant, g_{YM}^2, has dimension $3-d$ for Yang-Mills in d spatial dimensions. Therefore in five-dimensions, the theory becomes free at low-energies.

When the five-branes are compactified on a two-torus, we obtain a natural description of four-dimensional Yang-Mills. This theory has a Coulomb branch parametrized by the expectation values for the scalar fields. The theory at the origin of the moduli space is an interacting conformal field theory characterized by the dimensionless Yang-Mills coupling constant. The theory is believed to be S-dual [6, 7]. In this construction, S-duality is made manifest in terms of the symmetry group of the torus [3]. A matrix definition of the wrapped five-branes should therefore give a formulation of Yang-Mills with manifest S-duality.

Compactification on a three-torus gives three-dimensional Yang-Mills at low-energies. The coupling constant is again dimensionful and we are driven to a strong coupling fixed point in the infra-red. The Yang-Mills theory has a manifest $Spin(7)$ symmetry acting on the seven scalars and associated fermions. However, the superconformal fixed point actually has $Spin(8)$ global symmetry, and is believed to be interacting [8, 9, 5]. For the abelian case, the extra dimension can be viewed as arising from the scalar dual to the gauge-field in three-dimensions.

Lastly, compactification on a four-torus gives two-dimensional Yang-Mills. In the infra-red, we are again driven to a strong coupling fixed point. In this case, the resulting two-dimensional sigma model is believed to have a target space given by the classical moduli space of vacua [9, 10]. For gauge group $U(k)$, the moduli space is

$$\mathcal{M} = \frac{(\mathbb{R}^8)^k}{\mathbf{S}_k}, \tag{1.1}$$

and the infra-red theory is an orbifold conformal field theory. Note that in this case, the spectrum for the two-dimensional theory in DLCQ is discrete since there are no non-compact dimensions.

[1] To orient the reader with our conventions, note that $\mathcal{N}=4$ Yang-Mills in four-dimensions has sixteen real supersymmetries.

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While the goal of this discussion is to provide matrix descriptions for each of these theories, these Yang-Mills theories themselves define M theory compactifications in the DLCQ formalism. We can interpret each of the features described above in terms of some property required by compactified M theory. The two-dimensional Yang-Mills theory describes M theory on a circle. In the limit where the circle goes to zero size, the Yang-Mills coupling is driven to infinity and so the orbifold conformal field theory should describe the type IIA string with k units of longitudinal momentum [11, 9, 10]. Since the infra-red theory is an orbifold conformal field theory, the resulting Hilbert space agrees with the Hilbert space for a free string. Our matrix description of this theory can then be viewed as a matrix model for the matrix model of the type IIA string. Likewise, the type IIB string is defined in terms of the three-dimensional Yang-Mills theory, where the enhanced flavor symmetry reflects the appearance of an extra dimension needed for the ten-dimensional string theory. The existence of S-duality in four-dimensional Yang-Mills is needed for the T-duality symmetry which exists in M theory compactified on a three-torus [12, 13].

There are several different ways to obtain matrix formulations of Yang-Mills theories. One approach is to determine the matrix description of wrapped M theory five-branes. In a suitable limit, this matrix theory will describe the compactified $(2,0)$ field theory. This is an interesting question which involves a study of impurity theories similar to the system described in [14]. A second approach is to start with a non-critical string theory in six dimensions: either the $(2,0)$ string theory living on type IIA NS five-branes or the $(1,1)$ string theory living on type IIB NS five-branes [15]. On compactification, these two theories are part of a connected moduli space. The advantage of studying the string theory rather than the field theory is that there are two different matrix models for the compactified string theory. One is an impurity model which is quite difficult to analyze while the second is a more conventional gauge theory. There are related formulations which can be reached by a series of dualities from one of the above prescriptions, and we will explain one additional formulation. When restricted to energies $E \ll M_s$, where M_s is the string scale, these models have branches describing Yang-Mills fixed points.

In the following section, we describe how to obtain matrix models from some of these various approaches and the relations between them. Along the way, we will describe a duality between certain six-dimensional decoupled theories. The duality has the interesting feature of exchanging N with a different parameter and may shed some light on the large N limit of matrix theory. We will also find relations between the Coulomb branch of certain quiver gauge theories and degenerations of the moduli space of instantons on $\mathbf{T}^4$. In section three, we consider the conditions under which we can reduce to quantum mechanics and obtain a matrix formulation of the Yang-Mills fixed points. In section four, we explore the properties of supersymmetric Yang-Mills (SYM) from the matrix model description. We discuss electric and magnetic fluxes, Wilson lines in the longitudinal direction, vector particles, Coulomb interactions and R-symmetries. This includes a discussion of the perturbative limit of four-dimensional Yang-Mills where we show how the force between vector bosons can be reproduced in the matrix model. This is quite non-trivial since in a description where S-duality is manifest, the perturbative limit is not distinguished. Finally in section five, we conclude by discussing generalizations to cases with less supersymmetry, and by mentioning some of the many open questions. Some related issues have been discussed recently in [16].

2 Various approaches to compactified string theories

2.1 Compactified six-dimensional string theories

Four-dimensional $\mathcal{N}=4$ SYM can be viewed as a limiting case of a more general theory obtained by compactifying a 5+1D field theory on a torus, T^2. We could compactify either 5+1D SYM with $(1,1)$ supersymmetry or the $(2,0)$ field theory living on parallel five-branes. In the second case, the limit where we recover 3+1D SYM requires taking T^2 to zero size while holding fixed the complex structure τ. The coupling in the resulting SYM theory is τ and $SL(2,\mathbb{Z})$ is manifest. In turn, the $(2,0)$ or $(1,1)$ field theories describe the low-energy excitations of complete theories living on, respectively, type IIA or IIB NS five-branes [15]. We can therefore start with the more general question of how to provide a matrix model for the string theory compactified on $\mathbf{T}^d$ for $d=1,2,3,4$. The matrix model for SYM in D+1 dimensions can be deduced by taking a limit of the external parameters in the model for $d=5-D$.

Compactifying k NS five-branes on $\mathbf{T}^4$ gives an effective theory with sixteen supersymmetries which we will call $\mathcal{C}_k$. For a generic torus, $\mathcal{C}_k$ is 1+1D and contains all of the KK states of the compactified six-dimensional string theory. T-duality of the compactified string theory implies that $\mathcal{C}_k$ depends on an *external* parameter,

$$u \in SO(4,4,\mathbb{Z})\backslash SO(4,4,\mathbb{R})/(SO(4)\times SO(4)),$$

where u parametrizes the shape and size of $\mathbf{T}^4$ and the choice of $B_{\mu\nu}$. The matrix model for the $p_{\|}=N$ sector of $\mathcal{C}_k(u)$ is a theory $\mathcal{C}_{k,N}(u)$ with eight supersymmetries which depends on the external parameter u. Since $\mathcal{C}_k(u)$ has BPS strings, $\mathcal{C}_{k,N}(u)$ should generically be a 1+1D conformal field theory compactified on $\mathbf{S}^1$. There are many different but related ways to define $\mathcal{C}_{k,N}(u)$. We will discuss the following three ways of obtaining matrix formulations.

Route 1: We can study the theory localized at the singularity of M theory on $T^d\times A_{k-1}$. This geometric picture is particularly nice since it extends easily to the case of D and E singularities. When $d>3$, this theory needs to be defined in terms of the 5+1D string theories discussed in [17]. These string theories are obtained by placing N type IIB NS five-branes near an A_{k-1} singularity. The case of type IIA five-branes has not been considered but we will demonstrate a duality relating type IIA to type IIB. This duality is quite interesting since it exchanges N and k.

Route 2: We can consider the matrix definition of k longitudinal type IIA NS five-branes wrapped on $\mathbf{T}^4$. The matrix model for this case is defined in terms of the decoupled

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theory on N NS five-branes wrapped on the dual torus, $\widehat{T}^4 \times \widehat{S}^1$. We will study the sector with k units of string winding on $\widehat{S}^1$. This is the model obtained by probing wrapped NS five-branes in string theory.

Route 3: Lastly, we can take the theory living on k NS five-branes, wrapped on $\mathbf{T}^4 \times \mathbb{R}^{1,1}$, and view it as a compactification of a six-dimensional Lorentz invariant theory decoupled from gravity and string theory [15]. A description of $\mathcal{C}_{k,N}(u)$ can then be obtained by applying the $SO(5,5,\mathbb{Z})$ T-duality to the prescription given in [18, 19] which relates the DLCQ theory to the theory on a small space-like circle.

2.2 Route 1: From M-theory on an A_{k-1} singularity

The matrix model for M theory on $T^d \times$ ALE follows readily from the results in [20]. The radii of the torus are $\{R_1, \ldots, R_d\}$. Let us specialize to the case of $T^d \times A_{k-1}$ for the moment. The theory is $d+1$-dimensional Yang-Mills with gauge group,

$$U(N)_1 \times \ldots \times U(N)_k, \tag{2.1}$$

and eight supersymmetries. The hypermultiplet content is encoded in the extended Dynkin diagram. Each link gives a hypermultiplet in the representation, $\oplus_{ij}\, a_{ij}(N_i, \bar{N}_j)$, where a_{ij} is one for a link between the i^{th} and j^{th} node and zero otherwise. For A_{k-1}, this gives hypermultiplets in the $(N, \bar{N})$ of $U(N)_i \times U(N)_{i+1}$ for $i = 1 \ldots k$, where $k+1 \equiv 1$. The gauge group and matter content will actually change for $d \geq 3$ in a way which we will describe later. For $d > 3$, the gauge theory is no longer well-defined in the ultra-violet. The natural definition is in terms of the decoupled six-dimensional theory living on type II five-branes on the A_{k-1} singularity [17]. When all the Fayet-Iliopoulos parameters are set to zero, there is a Coulomb branch as well as a Higgs branch. The Coulomb branch describes the physics localized at the singularity, which is a $7-d$-dimensional gauge theory. The Higgs branch should describe the spacetime physics which is clearly not localized at the singularity. Our interest is primarily with the Coulomb branch in this discussion. The gauge theory associated to the singularity in M theory has a coupling constant,

$$(\hat{g}_d)^2 = \frac{1}{M_{pl}^3 R_1 \ldots R_d}. \tag{2.2}$$

The D and E cases are similar, differing only in the choice of gauge group and matter content.

What is the relation of M theory on $T^d \times A_{k-1}$ to compactified six-dimensional string theories? If R_1 is small, we can reduce from M theory to type IIA compactified on $T^{d-1} \times A_{k-1}$ with,

$$M_s^2 = M_{pl}^3 R_1. \tag{2.3}$$

The ten-dimensional type IIA string coupling is $g_s^A = M_s R_1$. When R_2 becomes small, we should T-dualize to type IIB with string coupling $g_s^B = R_1/R_2$. Describing the compactified string theory via M theory on an orbifold space can be directly related to compactified

NS five-branes by a T-duality argument. Let us start with k wrapped type IIB NS five-branes at a point on $\mathbb{R}^3 \times \mathbf{S}^1$ where the radius of the circle is T. T-dualizing on S^1 takes the NS five-branes to type IIA Kaluza-Klein monopoles with the scale of the metric set by $1/M_s^2 T$. In the limit where T becomes small, the Kaluza-Klein monopole degenerates to an A_{k-1} singularity [21, 22]. In this limit, we can use the matrix theory described in (2.1).

The k NS five-branes are then at a point on $\mathbb{R}^3 \times \mathbf{S}^1$ where the circle is very small. However, from the perspective of the gauge theory on the brane, the circle actually has radius $1/R_1$ [23]. We can make this circle large by taking R_1 small. The limit that we will want to study results in a theory parametrized by T^{d-1} and M_s, and is obtained by taking:

$$M_{pl} \to \infty, \qquad R_1 \to 0, \qquad T \to 0. \tag{2.4}$$

In this region of parameter space, the theory localized at the singularity will describe the compactified six-dimensional string theory. With M_s held fixed, it will interpolate between the compactified $(1,1)$ and $(2,0)$ string theories as we vary the parameters of T^{d-1}. Finally, considering energies much smaller than M_s gives the infra-red behavior of Yang-Mills compactified on T^{d-1}. We can therefore conclude that by studying the Coulomb branch of matrix theory on $T^d \times \mathrm{A_{k-1}}$, we will obtain matrix definitions of Yang-Mills in various dimensions.

2.3 Interpolating between six-dimensional string theories

The matrix theory torus is "dual" to the M theory torus. In terms of the longitudinal direction with size $R_{||}$, the torus has radii:

$$\begin{aligned} \Sigma_1 &= \frac{1}{M_{pl}^3 R_{||} R_1} = \frac{1}{M_s^2 R_{||}}, \\ \Sigma_j &= \frac{1}{M_{pl}^3 R_{||} R_j} = \frac{R_1}{M_s^2 R_{||} R_j}, \qquad j = 2 \ldots d. \end{aligned} \tag{2.5}$$

The case $d = 1$ corresponds to the decoupled theory on type IIB NS five-branes [14, 24]. In this case, the Coulomb and Higgs branches are expected to decouple in the infra-red [3]. As a first approximation to the physics of the Coulomb branch, we can use a moduli space approximation and describe the low-energy dynamics by a sigma model with a metric that has a "tube" structure. This is unlikely to be a good description of the physics at short distances. The failure of the moduli space approximation already occurs for M theory on an A_{k-1} singularity where $d = 0$. Quantum mechanics on the Coulomb branch can be approximated by a sigma model with a metric that behaves as $1/r^3$ at short distances in five-dimensions. However, the gauge bosons should appear as L^2 ground states in the quantum mechanics but these states generally cannot be seen without including more degrees of freedom. It seems likely that the Coulomb branch theory in $1+1$-dimensions also requires more degrees of freedom to be sensible.

Let us turn to $d = 2$ keeping M_s finite. In this and subsequent cases, there is no issue about the small distance behavior of the Coulomb branch metric since there is a genuine moduli space. How do we see T-duality? As $R_1 \to 0$, $\Sigma_2 \to 0$ so the theory becomes $1+1$-dimensional. The matrix model coupling is,

$$g^2 = \frac{R_{\|}}{R_1 R_2}. \tag{2.6}$$

The dynamics in this theory is governed by the dimensionless parameter,

$$\gamma = g^2 \Sigma_2 = \frac{1}{(R_2 M_s)^2}. \tag{2.7}$$

When $R_2 \to \infty$, $\gamma \ll 1$ so the effective gauge interactions are weak. We can then dimensionally reduce to two-dimensions and flow to the Coulomb branch conformal field theory. The Coulomb branch is parametrized by $4Nk$ scalars and describes the decoupled theory on type IIB NS five-branes.

When $R_2 \to 0$, $\gamma \gg 1$ so we first flow to the three-dimensional fixed point. Including the dual scalars, the Coulomb branch is again parametrized by $4Nk$ scalars, as we expect for a six-dimensional theory. The new direction emerges as the periods for the dual scalars decompactify. The Coulomb branch is quantum corrected but we can use a three-dimensional mirror to determine the metric [25, 26]. Including quantum corrections, the moduli space corresponds to the moduli space of N instantons in $SU(k)$ gauge theory. Reduction to two-dimensions then gives the matrix description of the decoupled theory on type IIA NS five-branes. This matrix description can also be obtained from the Higgs branch of the two-dimensional model with $U(N)$ gauge symmetry, an adjoint hypermultiplet and k fundamentals [27, 28]. This two-dimensional analogue of mirror symmetry is the simpler side of the duality described in [14], since it follows immediately from three-dimensional mirror symmetry. Note that in both limits, we see a six-dimensional theory on the Coulomb branch. In both the type IIA and type IIB cases, knowing the moduli space is unlikely to be sufficient to describe the IR limits of the Higgs or Coulomb branch theories. Without a better understanding of the IR limits, the matrix definition is still largely implicit.

2.4 A duality between decoupled theories

At this point, we cannot resist explaining an interesting duality between six-dimensional string theories. This duality was conjectured by Intriligator [17].[2] As we just mentioned, matrix models for SYM can be obtained by compactifying the six-dimensional string theories associated to type IIB five-branes at ADE singularities. Consistency conditions for these theories have been described in [29]. We could ask similar questions about type IIA five-branes at ADE singularities. On compactification, the theory of N type IIA five-branes at an ADE singularity is related by T-duality on a longitudinal circle to the IIB case but in six-dimensions, the theories are distinct and we obtain no equivalence this way. Let us take N IIB NS five-branes on an A_{k-1} singularity. We can equivalently

[2] We wish to thank K. Intriligator for bringing this to our attention.

consider N type IIB five-branes on a KK monopole with an A_{k-1} singularity. In the decoupling limit, as discussed above, the period for the scalar parametrizing the compact transverse circle decompactifies. We are then free to T-dualize on the compact direction.

Under this T-duality, the N IIB five-branes turn into a type IIA KK monopole with an A_{N-1} singularity. The type IIB KK monopole turns into k type IIA NS five-branes. With no additional compact directions, we are free to take the decoupling limit in both pictures. This gives an equivalence between the decoupled theory on N IIB five-branes at an A_{k-1} singularity and the decoupled theory on k type IIA five-branes at an A_{N-1} singularity.[3] In particular, this duality allows us to trade the usual large N limit of the matrix model, about which we know little, for a study of the matrix model with fixed P^+ but on a highly singular space. We might hope to gain some understanding of the large N limit from this approach and this is currently under investigation.

By including orientifold actions, this argument generalizes to the case of D singularities since we can replace a D singularity by an ALF space analogous to the multi-Taub-NUT metric associated to A singularities. We can also generalize it to the case of type IIB (p,q) five-branes at an A_{k-1} singularity by considering an M theory dual without a conventional type IIA description. The dual of a (p,q) five-brane described in [24] is M theory on $X_{p,q} = (\mathbb{C}^2 \times \mathbf{S}^1)/\mathbb{Z}_q$. In our case, the dual simply becomes k M theory five-branes on $X_{p,q}$. Since this leads us away from our main discussion, we will not explore further these theories and their generalizations here.

2.5 Route 2: Longitudinal five-branes in M theory

We can obtain another definition for $\mathcal{C}_{N,k}(u)$ and its low-energy limit describing the $(2,0)$ field theory on $\mathbf{T}^4$ by compactifying k parallel longitudinal type IIA or M theory five-branes. We will see that this route leads us to study the low-energy descriptions of gauge theories with point-like or string-like impurities in dimensions greater than four. For related comments see [31]. As explained in [14], interesting 1+1D theories can be obtained by inserting extra localized 1+1D degrees of freedom into a 2+1D theory with local gauge fields. The relevant question for us is how to elevate the impurity construction to cases where the bulk gauge theory is five or six-dimensional. We will define these bulk theories in terms of the decoupled theory on parallel NS five-branes wrapped on $\mathbf{S}^1 \times \widehat{\mathbf{T}}^4$ [15]. We will argue that in these cases, the impurities become dynamical in the sense that they can be constructed from *internal* degrees of freedom of the system. The dynamical impurity is in essence a generalization of the brane-probe technique. We apply this approach to 0-brane probes of the $(2,0)$ 5+1D CFT and 1-brane probes of the $(2,0)$ string theory.

We can obtain a matrix definition of five-branes wrapped on a torus, $\mathbf{T}^d$ with $d \leq 4$, by starting with the matrix model for a longitudinal five-brane [32] extended to k five-branes. This model contains a $U(N)$ vector multiplet X, an adjoint hypermultiplet H together with k hypermultiplets, Q^f, in the fundamental representation. The flavor index runs

[3] This exchange of N and k was noted independently in a geometrical construction of four-dimensional gauge theories [30].

from 1 to k. Let the worldvolume of the four-branes fill $x_1, \ldots, x_4$. The location of the four-branes is then specified by their positions in $x_5, \ldots, x_9$. We will only be concerned with the case where the four-branes are very close together, or actually coincident. It is interesting to note that the effective $U(k)$ gauge group appears as a *flavor* symmetry in the matrix description. This is particularly interesting when we consider large k color expansions.

The four scalars of the hypermultiplet H, denoted H^a for $a = 1 \ldots 4$, parameterize the position parallel to the longitudinal five-branes. We can derive the matrix model for 3+1D SYM from a limit of the $(2,0)$ theory compactified on $\mathbf{T}^4$ and the latter can be obtained from the low-energy limit of k parallel M theory five-branes [4]. Specifically, we take the limit where the distances between pairs of five-branes, $x_0^{(i)} - x_0^{(j)}$, goes to zero and where we restrict to energies of order $E \sim M_{pl}^{3/2} |x_0^{(i)} - x_0^{(j)}|^{1/2}$. In this way E^2 is of the order of the tension of BPS strings in the $(2,0)$ theory. The six-dimensional conformal field theory can be studied by taking the limit where the five-branes are coincident i.e. by setting all the positions $x_0^{(i)}$ to zero.

To find the matrix description for five-branes wrapped on a circle with radius R_1, we take the original system as well as all its translates along x_1. The gauge group becomes infinite-dimensional and we then quotient out by the symmetry group generating discrete translations along x_1 by multiples of $2\pi R_1$ [33, 1, 13]. In practice, we take a large gauge transformation, $\Omega \in U(\infty)$, and restrict to matrices obeying,

$$\begin{array}{lll} H_1 = \Omega^{-1} H_1 \Omega + 2\pi R_1, & H_a = \Omega^{-1} H_a \Omega, & a = 2 \ldots 4, \\ Q^f = \Lambda_{f'}{}^f \Omega^{-1} Q^{f'}, & X_\mu = \Omega^{-1} X_\mu \Omega, & \mu = 5 \ldots 9, \end{array} \tag{2.8}$$

where Λ is some fixed $k \times k$ flavor matrix. This matrix encodes the choice of Wilson line along x_1 for the gauge-field on the parallel four-branes. These constraints are solved by taking $X_1 = -i\partial_1 - A_1$, and by promoting the rest of the X and H variables to fields depending on the periodic coodinate x_1. In this representation, Ω is diagonal with eigenvalues e^{ix_1/R_1}. However, the constraint on the hypermultiplets is solved by localizing the hypermultiplets to k specific points $x_1 = \lambda_i$ where $i = 1, \ldots, k$ and where λ_i is an eigenvalue of the matrix Λ. Thus the system becomes a bulk 1+1D SYM with sixteen supersymmetries and with k impurities at fixed positions λ_i (see also [14]). The impurities explicitly break half the supersymmetries.

A similar argument applies when the five-branes are wrapped on $\mathbf{T}^d$. The bulk physics is then described by a $d+1$-dimensional $U(N)$ gauge theory with a $d+1$-dimensional adjoint hypermultiplet. The hypermultiplets in the fundamental representation are quantum mechanical and live at specific points on $\hat{\mathbf{T}}^d$. These hypermultiplets treat the spatial components of the connection, A_μ, as scalars. In particular, the coupling of the hypermultiplets to the fields $A_1, \ldots, A_d$ breaks Lorentz invariance. By choosing appropriate limits for the size of the compact directions, we should be able to recover a complete description of longitudinal p-branes with $p = 1, 2, 3$ and 4. This description should capture the spacetime metric as well as the physics localized on the brane itself. In this discussion, we are only concerned with the theory on the five-branes.

When we further compactify on $\mathbf{T}^4$ we naively find 4+1D SYM with 0+1D impurities. According to [34] we know that 4+1D SYM should be interpreted as the $(2,0)$ 5+1D theory compactified on $\mathbf{S}^1 \times \widehat{\mathbf{T}}^4$. How should the impurity be interpreted?

We claim that the impurity is no longer external but is defined as the sector with k units of KK momentum along $\mathbf{S}^1$. In this way we also make contact with the description of longitudinal objects as momentum states in the matrix model (see [35]). To justify the claim that at low energies k KK states look point-like we can estimate the various energy scales involved. We are looking for an appropriate description of the states at energy levels:

$$E = \frac{2\pi k}{r} + \epsilon, \qquad \epsilon \ll \frac{1}{r},$$

where r is the radius of the circle $\mathbf{S}^1$. The low-energy theory in 4+1D without the momentum has a moduli space of $(\mathbb{R}^5)^k/S_k$. Even at the origin of the moduli space the theory is free in the infra-red. At a generic point in the moduli space with a vacuum expectation value (VEV), $v = \sqrt{\sum_1^5 |\phi_i|^2}$, there are W bosons of mass $r^{1/2}v$ and also monopoles which are strings with tension $r^{-1/2}v$. The energy scale set by the strings is higher by a factor of $r^{-3/4}v^{-1/2}$. As long as $v \ll r^{-1}$ both scales are much smaller than the compactification scale. States with energies which satisfy,

$$E \ll r^{-1}, \tag{2.9}$$

can be described by the effective (non-renormalizable) Lagrangian of 4+1D SYM with dimensionful coupling constant,

$$\frac{1}{g_{\mathrm{YM}}^2} = r^{-1}.$$

Let us add one unit of momentum along the small $\mathbf{S}^1$, and ask what it looks like in the low-energy limit in 4+1D. The momentum state becomes a heavy soliton which we locate at the origin. It breaks half the supersymmetries. The original symmetries were the $SO(4,1)_N$ Lorenz invariance and $Sp(2)_R$ R-symmetry. The soliton breaks $SO(4,1)_N$ down to $SO(4)_N$ while $Sp(2)_R$ remains intact. We can realize this as a D0-brane on a D4-brane, making $Sp(2) \sim SO(5)$ geometrical. The unbroken supersymmetries transform as (see also [27]):

$$(\mathbf{2},\mathbf{1})_N \otimes \mathbf{4}_R$$

with a reality condition. An anti-0-brane will transform as

$$(\mathbf{1},\mathbf{2})_N \otimes \mathbf{4}_R.$$

At a generic point in moduli space, the soliton is described by an effective quantum mechanics. The effective quantum mechanics contains four bosons X^i and eight fermions. The bosons are in the $(\mathbf{2},\mathbf{2})_N \otimes \mathbf{1}_R$ while the fermions are in the $(\mathbf{1},\mathbf{2})_N \otimes \mathbf{4}_R$ with a reality condition.

At the origin of moduli space the KK state behaves like an instanton of 4+1D SYM. The low-lying states which are related to the impurity can be treated by quantizing the instanton moduli space. Thus the 0+1D theory has four more modes – the size and orientation – which parameterize the moduli space $\mathbb{R}^4/\mathbb{Z}_2$.

When the instanton has a finite size, it is no longer obvious that the low-energy description in terms of a 0+1D quantum mechanics interacting with a 4+1D bulk is still adequate. We can justify it in the following way. Suppose we take a small UV cutoff ϵ on the energies. The characteristic length scales of the IR processes we are describing will be ϵ^{-1}. To justify the 0+1D picture, we have to show that we can make a small wave-packet in instanton moduli space that will localize on instanton sizes $\rho \ll \epsilon^{-1}$; yet the energy spread of the wave-packet must be much smaller than ϵ. The Hamiltonian for the collective modes coordinatizing instanton moduli space is approximately,

$$H \sim r^{-1}\{\dot{\rho}^2 + \rho^2 \mathrm{tr}(\Omega^{-1}\dot{\Omega})^2\}.$$

where ρ is the size of the instanton and $\Omega \in SU(2)/\mathbb{Z}_2$ is the orientation. The low energy levels where ϵ is above $1/r$ correspond to wave functions ψ which behave like

$$\psi \sim e^{i(\epsilon/r)^{1/2}\rho}.$$

Thus the typical scale for ρ is,

$$\rho \sim \epsilon^{-1/2} r^{1/2},$$

and indeed, the bulk excitations with energy ϵ will have an approximate size of $\epsilon^{-1} \gg \rho$. This means that at energies $\epsilon \ll r^{-1}$, it is safe to assume that the instanton is point-like.

After compactification on $\mathbf{T}^4$ the previous argument can no longer be applied because the instanton is no longer localized. However, we will show that the energy scale of the instanton moduli space is much below the energy scale of momentum excitations.

Let us compactify the $(2,0)$ theory on $\mathbf{S}^1 \times \widehat{\mathbf{T}}^4$ with $\mathbf{S}^1$ of radius r as before and with $\widehat{\mathbf{T}}^4$ of radii $\Sigma_1, \ldots, \Sigma_4$ and take the limit $r \ll \Sigma_i$. The energy scale of the instantons themselves is,

$$M_{\text{instantons}} \sim \frac{1}{g^2} \sim r^{-1}.$$

The other momentum states are at mass scale

$$M_{\text{momentum}} \sim \Sigma_i^{-1} \ll r^{-1}.$$

The energy of an electric flux in the direction of Σ_1 is given by

$$M_{\text{electric}} \sim \frac{r\Sigma_1}{\Sigma_2\Sigma_3\Sigma_4} \ll \Sigma_1^{-1},$$

and a magnetic flux in directions Σ_1, Σ_2 has energy

$$M_{\text{magnetic}} \sim \frac{\Sigma_3\Sigma_4}{r\Sigma_1\Sigma_2} \sim r^{-1}.$$

In the regime $r \ll \Sigma_i$, the low-energy is dominated by the 0+1D quantum mechanics and the electric fluxes. We can see that the energies coming from the 0+1D quantum mechanics are of the same order of magnitude as the electric fluxes. We can check this for the zero modes X^i which represent the center of mass of the instanton. The momentum of the instanton is quantized in units of $1/\Sigma_i$. Since its mass is r^{-1}, the kinetic energy will be of the order of r/Σ_i^2. This is to be expected since the electric fluxes are obtained by quantizing the global Wilson lines along $\widehat{\mathbf{T}}^4$ which are part of the instanton moduli space.

We can then conclude that the matrix model for the $(2,0)$ theory compactified on $\mathbf{T}^4$ is given by quantum mechanics on the moduli space of $U(N)$ instantons with instanton number k on $\widehat{\mathbf{T}}^4$.

2.6 Longitudinal five-branes in string theory

Starting with the previous construction of k M-theory five-branes wrapped on $\mathbf{T}^4$, we can further compactify a direction transverse to the five-branes. This way we obtain k type IIA NS five-branes wrapped on $\mathbf{T}^4$. The matrix formulation looks naively like 5+1D SYM compactified on $\widehat{\mathbf{T}}^5$ with k 1+1D lines of impurities located parallel to one side of $\mathbf{T}^5$. Let us take the sides of $\widehat{\mathbf{T}}^5$ to be $\Sigma_1, \ldots, \Sigma_5$ and let the k impurities be parallel to Σ_5. The bulk 5+1D SYM has to be defined using the theory on N type IIB five-branes with the $SO(5,5,\mathbb{Z})$ T-duality group [15]. This theory has string-like BPS excitations and so we interpret the 5+1D SYM with impurities as the sector with k units of winding number along Σ_5. The impurities are dynamical as in the previous case. Applying $SO(5,5,\mathbb{Z})$ T-duality, we can map the sector with k units of winding number to a sector with k units of momentum along a dual $\widetilde{\Sigma}_5$. In this way, we again make contact with the identification of longitudinal objects and KK momentum in the matrix model.

Now let us take the limit $\Sigma_5 \to \infty$, keeping $\Sigma_1, \ldots, \Sigma_4$ fixed and look for a low-energy description of the resulting 1+1D theory. To be more generic we can put the string theory on $\mathbf{T}^4 \times \mathbb{R}^{1,1}$ with k strings stretched along the uncompactified direction. We denote this 1+1D theory by $\mathcal{C}'_{k,N}(u)$. It depends via u on the sixteen external parameters which parameterize,

$$\mathcal{M}' = SO(4,4,\mathbb{Z})\backslash SO(4,4,\mathbb{R})/(SO(4)\times SO(4)).$$

In the spacetime interpretation of the matrix model, the limit $\Sigma_5 \to \infty$ corresponds to taking the type IIA string coupling $g_s \to 0$. Thus $\mathcal{C}'_{k,N}$ is a matrix model for k wrapped NS5-branes in type IIA at $g_s = 0$ compactified on $\widehat{\mathbf{T}}^4$.

2.7 Route 3: From Lorentz invariance

We could also derive the 1+1D theory $\mathcal{C}_{k,N}(u)$ by a straightforward application of the rules of [19]. To find the sector $p_{\|} = N$ of $\mathcal{C}_k(u)$ in the DLCQ, we can consider the string theory on k five-branes with parameters,

$$\widetilde{M_s}, \widetilde{R_i}, R_s \tag{2.10}$$

and N units of KK momentum along R_s. This theory is related to the $\mathcal{C}_{k,N}(u)$ by a Lorentz transformation. We can fix the parameters (2.10) in terms of M_s, the original transverse dimensions R_i and the longitudinal direction $R_{\|}$:

$$R_s \widetilde{M_s}^2 = M_s^2 R_{\|}.$$

As $R_s \to 0$, the radii $\widetilde{R_i} \to 0$ and $\widetilde{M_s} \to \infty$. Using the $SO(5,5,\mathbb{Z})$ T-duality, we can exchange the shrinking circle R_s for a circle with finite radius, $1/M_s^2 R_{\|}$. The momentum is exchanged for N units of string winding. This leads to a matrix formulation in terms of k NS five-branes wrapped on $\mathbf{T}^4$ with sides $\widetilde{R_i}$. The parameters for this torus are again determined by the choice of u. We have performed a single T-duality to arrive at this description, but $\mathcal{C}_{k,N}(u)$ has two low-energy limits corresponding to the theory on either type IIA or type IIB five-branes. It is then convenient to T-dualize again on one of the transverse circles which sends,

$$u \to T(u),$$

where the map T is independent of our choice of transverse circle. We can define the map by picking an element, $t \in O(4,\mathbb{Z}) \times O(4,\mathbb{Z})$, with $\det(t) = -1$. Then $T(u) = [t \circ g \circ t^{-1}]$ where $g \in SO(4,4,\mathbb{R})$ is any representative for u. This construction is independent of the choice of t. We will need this map to relate this proposal to the previous ones.

2.8 Relations between the various approaches

We have obtained three seemingly different matrix formulations for the compactified six-dimensional string theory. How are they related? Let us start by considering the matrix model describing energies $E \ll M_s$ which corresponds to the $(2,0)$ field theory on $\mathbf{T}^4$. We have seen that route (2) leads to quantum mechanics on the moduli space of $U(N)$ instantons on the dual $\widehat{\mathbf{T}}^4$ at instanton number k, while route (3) leads to $U(k)$ instantons on the original $\mathbf{T}^4$ at instanton number N. In this case, the equivalence follows from the Fourier-Mukai transform [36], or from T-duality of a system of N 4-branes and k 0-branes on $\mathbf{T}^4$.

In fact, we have a more general statement that $\mathcal{C}_{k,N}(u)$ is equivalent to $\mathcal{C}_{N,k}(T(u))$ which implies the above relation at low-energies. This equivalence follows from the T-duality argument of section *2.3* and proves that route (2) and route (3) give equivalent formulations. By contrast, route (1) leads to a matrix description in terms of the Coulomb branch for the Yang-Mills theories (2.1) in various dimensions. The Coulomb branch for the theory of N NS five-branes wrapped on T^3 at an A_{k-1} singularity is a hyperKähler manifold. This compactified string theory is parametrized by

$$SO(3,3,\mathbb{Z}) \backslash SO(3,3,\mathbb{R}) / (SO(3) \times SO(3)),$$

which is

$$\sim SL(4,\mathbb{Z}) \backslash SL(4,\mathbb{R}) / SO(4).$$

So the moduli space is parametrized by the shape of a $\mathbf{T}^4$. The equivalence of this approach with routes (2) and (3) implies that this is the moduli space of $U(k)$ instantons on $\mathbf{T}^4$ with instanton number N.[4] In the next section, we will see that a particular degeneration of this hyperKähler space does indeed agree with the solution of 3+1D $\mathcal{N}=2$ quiver gauge theories found by other approaches [37, 30].[5]

2.9 (p,q) Five-brane theories

We can similarly study the matrix model for the (p,q) five-brane theory compactified on $\mathbf{T}^d$. We will again go along the three routes described above. We will see that in each route, we can identify a certain $\mathbb{Z}_q$ global symmetry of $\mathcal{C}_{q,N}$ and twisting by the symmetry defines the (p,q) five-brane theory. Let us first discuss the low-energy limit in $(6-d)$ dimensions.

For a generic $\mathbf{T}^d$, the theory has $(6-d)$ uncompactified dimensions and depends on d^2 external parameters,

$$u \in SO(d,d,\mathbb{Z})\backslash SO(d,d,\mathbb{R})/(SO(d)\times SO(d)),$$

which are the shape, size, and B-fields of $\mathbf{T}^d$. Let

$$r = \gcd(p,q).$$

As explained in [24], in 5+1D the low-energy limit is given by $U(r)$ SYM, which at a generic point in the moduli space is described by r vector multiplets. This is also true after compactification. In particular for $d=2$, we have a 3+1D theory and the low-energy modes are generically r free vector multiplets. At the origin of moduli space, the low-energy theory is the interacting theory to which $U(r)$ SYM flows in the IR, with a coupling constant given by the area of $\mathbf{T}^2$ times M_s^2. This coupling constant is defined up to S-duality. Thus for generic $\mathbf{T}^d$, theories with different (p,q) but equal r flow to the same IR theory. This is similar to the situation in 5+1D where p and q entered only through an irrelevant operator of dimension 6, i.e. $\frac{p}{q}\mathrm{Tr}F^3$.

For special values of u the story is different. Just like the NS5-brane theory, the (p,q) theory has several special limits for u where a maximal number of dimensions decompactify making the theory six-dimensional. One such limit is when $\mathbf{T}^d$ becomes large and we are back to the uncompactified (p,q) five-brane theory. Other limits are obtained when $\mathbf{T}^d$ is in the form $\mathbf{T}^{d-d'}\times\mathbf{T}^{d'}$ with $\mathbf{T}^{d'}$ small and $\mathbf{T}^{d-d'}$ large. To find the low-energy description in these limits, we have to study how the (p,q) theory behaves under T-duality. This is easy to analyze using the realization of the theory in terms of bound states given in [15]. In this way, we can identify p with a D5-brane charge. After T-duality, we then get a system of NS five-branes with some D-brane charges. The results are as follows: for $d=1$, we start with the (p,q) five-brane compactified on a very small $\mathbf{S}^1$. This can be dualized to q type IIA NS5-branes on a large $\widehat{\mathbf{S}}^1$. At low-energies, this

[4] That the Coulomb branch is given by the moduli space of instantons on T^4 was suggested by probe arguments in [17].

[5] Some related observations about theories with compact moduli spaces have been made independently by [38].

theory contains q tensor multiplets with compact scalars, and the p charge is observed at low-energy energies as a non-zero gradient for the compact scalars. For $d = 2$, we start with the (p, q) five-brane compactified on a very small $\mathbf{T}^2$. This can be dualized to the $(0, q)$ five-brane on a very large $\widehat{\mathbf{T}}^2$. The p charge becomes magnetic flux along this $\widehat{\mathbf{T}}^2$. Note that this is consistent with the fact that p takes values in $\mathbb{Z}/q\mathbb{Z}$, i.e. it is defined modulo q. A magnetic flux which is a multiple of q can be embedded entirely within the overall $U(1)$ factor of $U(q)$ and decouples from $SU(q)$. Similarly, for $d = 4$ we find the $(0, q)$ five-brane on a large $\widehat{\mathbf{T}}^4$ with p units of electric flux transverse to $\widehat{\mathbf{T}}^4$. For $d = 3$ we find type IIA five-branes with some form of partial tensor flux. This suggests that the $(2, 0)$ theory also has a tensor flux in $\mathbb{Z}/q\mathbb{Z}$.

As a consequence of this discussion, we can argue that the spectrum of the theories is not continous in the parameter $\theta = 2\pi\frac{p}{q}$. Recall that the coefficient of the term $\mathrm{tr}\{F^3\}$ in the derivative expansion of the low-energy description of the (p, q) theory is θ as argued in [24]. The question then arose about whether the full (p, q) theory is continuous in θ. See also the discussion in [39]. The discussion above suggests that this is not the case, at least for the compactified theories. The Hamiltonian of the (p, q) theory on $\mathbf{T}^5$ depends on 25 parameters which parameterize $\mathbf{T}^5$ and the B-fields. The uncompactified theory is reached in a certain limit of these parameters. As we have seen, there are other low-energy limits obtained by shrinking some sides of the $\mathbf{T}^5$ to zero and using T-duality. We can use an element of $SO(5, 5, \mathbb{Z})$ which maps the p flux into a magnetic flux and in this way, we reach another low-energy limit which has the full $U(q)$ SYM gauge group. Thus given a 5+1D theory, we might define the integer q as the maximal rank of the gauge groups attained in any of the possible low-energy limits of the theory.

The analog of route (1) is to start with the geometrical dual of the (p, q) five-brane [24]. This is given in terms of M theory on $X_{p,q} = (\mathbb{R}^4 \times \mathbf{S}^1)/\mathbb{Z}_q$ where the $\mathbb{Z}_q$ acts on $\mathbb{R}^4$ to give an A_{q+1} singularity and on $\mathbf{S}^1$ by a shift. The decoupling limit requires taking the size of $\mathbf{S}^1$ to zero. The matrix model for this M theory background follows immediately from studying zero-brane dyanmics on $X_{p,q}$ [24]. It is a 1+1D gauge theory on $\widehat{\mathbf{S}}^1$ with gauge group,

$$U(N)_1 \times \cdots \times U(N)_q,$$

and cyclic hypermultiplets. The integer p enters as a twisted boundary condition along $\widehat{\mathbf{S}}^1$ for the global $\mathbb{Z}_q$ symmetry which rotates the chain. To compactify on $\mathbf{T}^d$, we take a $(d + 2)$-dimensional gauge theory with the same field content compactified on $\widehat{\mathbf{T}}^d \times \widehat{\mathbf{S}}^1$, with boundary conditions twisted by the global $\mathbb{Z}_q$ along $\widehat{\mathbf{S}}^1$. For sufficiently large d, the description in terms of a field theory will break down and we should look for a decoupled theory to define the matrix model. In this case, the desired string theories are easily related to [17]. However, in other cases, for example those related to non-simply-laced gauge groups, there are theories with novel properties. We hope to report on these models elsewhere.

In route (2), we start with M theory compactified on $\mathbf{T}^2$ and add a longitudinal five-brane wrapping a (p, q) 1-cycle. The matrix model for this case is given by 2+1D SYM compactified on $\widehat{\mathbf{T}}^2$ with a line of impurities along a (p, q) 1-cycle. In a fundamental cell, this looks like $\frac{pq}{r}$ parallel lines of impurities. After compactification on a further $\widehat{\mathbf{T}}^3$, we obtain what at first sight looks like the $(0, q)$ theory compactified on $\widehat{\mathbf{T}}^4 \times \mathbf{S}^1$ with $\frac{pq}{r}$ parallel lines of impurities. As in the previous discussion, we are taking the limit where $\mathbf{S}^1$ is much bigger than $\widehat{\mathbf{T}}^4$. The phrase "lines of impurities" has to be translated into the phrase "the sector with q units of string winding along $\mathbf{S}^1$ and p units of winding along one of the other directions of $\widehat{\mathbf{T}}^4$". In terms of the previously defined 1+1D theory $\mathcal{C}_{q,N}$, this is interpreted as follows. $\mathcal{C}_{q,N}$ has 8 conserved quantum numbers associated to momentum and winding along $\mathbf{T}^4$. Each of them is related to a $U(1)$ symmetry corresponding to a shift in an approprite direction in $\mathbf{T}^4$, or after T-duality $\widehat{\mathbf{T}}^4$. We pick a $\mathbb{Z}_q$ subgroup of one of these $U(1)$ symmetries and take $X_{q,N}$ on $\mathbf{S}^1$ with a twisted boundary condition with respect to this subgroup. Heuristically, the effect of this twist is to close the q strings along $\mathbf{S}^1$ up to a shift, which exactly agrees with the impurity picture.

In route (3) we start with the (p, q) theory compactified on a small $\mathbf{S}^1$ with N units of KK momentum. We then perform T-duality along this $\mathbf{S}^1$ and another direction of the $\mathbf{T}^4$. We end up with the $(0, q)$ five-brane theory compactified on another $\mathbf{T}^4$ with q units of winding as before. The new ingredient is that the p charge becomes magnetic flux along the $\widehat{\mathbf{S}}^1$ and another direction of $\mathbf{T}^4$. We see again that the (p, q) theory is related to twisting by an element of a global $\mathbb{Z}_q$ symmetry of $\mathcal{C}_{N,q}$. This time the $\mathbb{Z}_q$ symmetry in question is a gauge transformation by a topologically nontrivial gauge transformation of $SU(q)$. This gauge transformation corresponds to a nontrivial generator of $\pi_1(SU(q))$ [40].

3 Matrix models for Yang-Mills theories

We are now ready to take the IR limit in various dimensions to obtain matrix models for SYM theories. We start with M theory on $T^d \times \mathrm{A}_{\mathrm{k}-1}$ where the radii of the torus are $\{R_1, \ldots, R_d\}$. The effective SYM coupling constant is given in (2.2). The matrix model, given in section *2.2*, has eight supersymmetries and lives on the dual torus, $\widehat{\mathbf{T}}^d$, with sides

$$\Sigma_j = \frac{1}{M_{pl}^3 R_{\|} R_j}.$$

The matrix model coupling is,

$$(g_{d+1})^2 = \frac{1}{M_{pl}^{3(d-2)} {R_{\|}}^{d-3} R_1 R_2 \ldots R_d}, \tag{3.1}$$

and for $d < 4$, the matrix model is a gauge theory. The scale of the matrix theory is set by the effective coupling constant and for $d = 3$, the theory is conformal.

We will show below that the limit in which the matrix theory describes Yang-Mills corresponds to reducing the degrees of freedom of the full matrix model to quantum mechanics on the Coulomb branch. We restrict ourselves to energies below the compactification scale,

$$E \ll R_1^{-1}, \ldots, R_d^{-1}, \tag{3.2}$$

with the spacetime coupling $\hat{g}_d$ held fixed. A word on notation: we will occasionally take a limit like,

$$R_j \to 0,$$

which is shorthand for "we keep our energy scale much smaller than R_j^{-1}". There are now two issues that we need to address: the first is whether the energy scale set by the sides Σ_j of $\mathbf{T}^d$ is small compared to the scale set by the coupling. If so, we can first solve for a desciption of the low-energy physics using the dynamics of $d+1$-dimensional SYM. The second issue is whether the energy scale we wish to observe is smaller than Σ_j^{-1}, in which case, we can further reduce to quantum mechanics on the moduli space. We will consider this reduction case by case.

3.1 Five-dimensional SYM

To obtain 4+1D SYM, we consider M theory on $\mathbf{T}^2 \times A_{k-1}$. As we discussed in the previous section, this can be viewed as compactifying the $(1,1)$ string theory on a circle. The classical moduli space can be parametrized by Wilson lines along the compact circles,

$$\mathcal{M} = \left[(\mathbb{R}^3 \times T^2)^N / \mathbf{S}_N\right]^k, \tag{3.3}$$

while the metric generally receives quantum corrections. The dynamics is governed by the parameters,

$$\begin{aligned} (g_3)^2\Sigma_1 &= \frac{1}{M_{pl}^3 R_1^2 R_2} = \frac{(\hat{g}_5)^2}{R_1} \gg 1 \\ (g_3)^2\Sigma_2 &= \frac{1}{M_{pl}^3 R_1 R_2^2} = \frac{(\hat{g}_5)^2}{R_2} \gg 1. \end{aligned}$$

We can therefore reduce to the 2+1-dimensional moduli space parametrized by the Wilson lines and the scalars in the vector multiplets. Restricting to energies obeying (3.2) and taking $M_{pl} \to \infty$ implies that our energy scale satisfies,

$$E \ll \frac{1}{\Sigma_1}, \frac{1}{\Sigma_2}.$$

We can therefore neglect the dependence on the coordinates x_1, x_2 and reduce to quantum mechanics on the moduli space.

The coupling in the quantum mechanics,

$$g_{QM}^2 = {R_\parallel}^3 M_{pl}^6, \tag{3.4}$$

goes to infinity. There are two energy scales associated to wavefunctions on (3.3) with non-constant dependence on the two compact directions. The energy scale for the direction coming from a Wilson line along x_1 is,

$$E \sim \frac{R_\parallel}{R_1^2} \to \infty.$$

We can then take our wavefunctions to be locally independent of the coordinate coming from A_1. Likewise, the other compact direction has an energy scale,

$$E \sim \frac{R_{||}}{R_2^2},$$

which becomes large as we take the five-dimensional limit, $R_2 \to 0$. We can then restrict to ground state wavefunctions on the compact directions, with a non-trivial dependence on the three non-compact directions. This limit describes the infra-red dynamics of five-dimensional Yang-Mills.

There is a second way to obtain this result which is essentially T-dual to the previous description and closely related to the field theory limit of our discussion in section *2.2*. We can define 4+1D SYM as the $(2,0)$ theory compactified on $\mathbf{S}^1$ [34]. The size of the circle is $\hat{g}_5^2$. In this case, we parametrize the Coulomb branch in terms of the scalars dual to the photons and the three scalars in a vector multiplet. Including quantum corrections, the Coulomb branch is a hyperKähler metric on $\left[\mathbb{R}^3 \times S^1)^N/\mathbf{S}_N\right]^k$ where the circles have size $\hat{g}_5^2$. In the limit $\hat{g}_5^2 \to \infty$, the manifold reduces to the moduli space of N instantons in $SU(k)$ gauge theory, and we recover our description of the $(2,0)$ field theory.

We then have a $2+1$-dimensional sigma model from $\widehat{\mathbf{T}}^2$ to the Coulomb branch. We would like to see the full compactified $(2,0)$ theory so we need to consider energies associated to the dual photons,

$$E \sim R_{||} M_{pl}^6 (R_1 R_2)^2.$$

The scale of fluctuations for the non-compact scalars, $1/\Sigma_i$, is much larger than the energies under consideration. The same is true for the scale of fluctuations for the dual photons for the same reason that we could neglect Wilson lines in the preceeding discussion. We can therefore restrict to quantum mechanics on the hyperKähler moduli space. In the IR limit, we can restrict to ground state wavefunctions along the compact directions, which means that the wavefunctions are locally constant in the compact variables.

Since the spectrum includes massless vector bosons, we should have L^2 zero-energy ground states in the quantum mechanical sigma model for every N. With this amount of supersymmetry, the ground states should correspond to elements of de Rham cohomology. For the case of $N=1$, it should be possible to check the existence of these forms using the metric presented in [41]. For example, for $k=2$, the metric has an A_1 singularity and the desired L^2 form comes from this singularity. Similar forms should exist for every N and k.

3.2 Four-dimensional SYM

The physics is considerably different for $d=3$. The $U(1)$ part of the $U(N)$ factors in (2.1) are not asymptotically free. The effective gauge group in the infra-red is then,

$$SU(N) \times \ldots \times SU(N). \tag{3.5}$$

Note that for $d < 3$, we require the gauge group given in (2.1) or for example, T-duality of the string theory would fail. The matrix theory in four-dimensions is then a finite theory with coupling,

$$g_4^2 = \frac{1}{M_{pl}^2 R_1 R_2 R_3}. \tag{3.6}$$

It is a requirement for this proposal to be sensible that the coupling be a true modulus of the theory. If the matrix theory were not a finite gauge theory, it almost certainly could not be describing an S-dual theory. This further supports the choice (3.5). For this reason, the matrix models for the D_k and E_6, E_7, E_8 cases must also be finite theories, and indeed they are finite. Again, the $U(1)$ factors and associated charged matter are frozen out.

The classical moduli space is now,

$$\mathcal{M} = \left[(\mathbb{R}^2 \times T^3)^{N-1} / \mathbf{S}_{N-1} \right]^k. \tag{3.7}$$

The spacetime coupling and the matrix model coupling are identical, and we will hold (3.6) fixed. Because the coupling is dimensionless, the dynamics is independent of the size of $\mathbf{T}^3$. At sufficiently low energies, we can always reduce to quantum mechanics on the moduli space. By low energies, we mean energies far below Σ_j^{-1}. To see this let us again estimate the relevant energy scales. Let us normalize the Lagrangian for a scalar field ϕ on the moduli space to be,

$$L = \int d^3x dt |\partial_\mu \phi|^2.$$

Truncating to quantum mechanics keeps only the zero mode for ϕ giving,

$$L = \Sigma_1 \Sigma_2 \Sigma_3 \int dt |\dot{\phi}|^2.$$

For energies $E \ll \Sigma_j^{-1}$, we can estimate the spread of the wavefunction $\psi(\phi)$ to be,

$$\Delta\phi \sim E^{-1/2} (\Sigma_1 \Sigma_2 \Sigma_3)^{-1/2}.$$

On dimensional grounds, higher derivative terms in the low-energy expansion will have more powers of $\phi^{-1}\partial_\mu$. We can estimate the size of these corrections:

$$\phi^{-1}\partial_j \sim E^{1/2} \left(\frac{\Sigma_1 \Sigma_2 \Sigma_3}{\Sigma_j^2} \right)^{1/2} \sim (E\Sigma_j)^{1/2} \ll 1.$$

We can then restrict to ground state wavefunctions on the torus, so finally restrict to quantum mechanics on the Coulomb branch of the quiver gauge theory.

It is interesting to derive this result from a limit of the $(2,0)$ theory compactified on $\mathbf{T}^2 \times \widetilde{\mathbf{T}}^2$. We take the limit where the complex structure of $\widetilde{\mathbf{T}}^2$ is fixed to be τ, and where the volume goes to zero:

$$\mathrm{vol}(\widetilde{\mathbf{T}}^2) \longrightarrow 0. \tag{3.8}$$

The other $\mathbf{T}^2$ is taken to be the transverse space for $\mathcal{N} = 4$ SYM. The matrix model for this theory is described by quantum mechanics on the moduli space of $U(N)$ instantons on the dual $\widehat{\mathbf{T}}^2 \times \hat{\widetilde{\mathbf{T}}}^2$ at instanton number k. This moduli space has $4Nk$ real dimensions. In the limit (3.8) this moduli space has large and small directions. Quantization of the motion along the small directions gives a very high energy scale which is related to the compactification scale of $\widetilde{\mathbf{T}}^2$. To get 3+1D SYM, we need to be far below this scale. This means that we are only interested in wavefunctions which do not vary locally along the small directions of the moduli space. There are $2Nk + 2$ large directions and $2Nk - 2$ small ones. To see this we note that in the limit where $\widehat{\mathbf{T}}^2$ is much smaller than $\hat{\widetilde{\mathbf{T}}}^2$, instantons on $\widehat{\mathbf{T}}^2 \times \hat{\widetilde{\mathbf{T}}}^2$ are characterized by specifying the $U(N)$ holonomies on the small $\widehat{\mathbf{T}}^2$. These are given by N points on the dual of $\widehat{\mathbf{T}}^2$ which brings us back to $\mathbf{T}^2$. This divisor of N points on $\mathbf{T}^2$ has to vary holomorphically over $\hat{\widetilde{\mathbf{T}}}^2$. It traces a Riemann surface $\Sigma \subset \mathbf{T}^2 \times \hat{\widetilde{\mathbf{T}}}^2$. It is an N-fold cover of $\hat{\widetilde{\mathbf{T}}}^2$ and it is also easy to see that Σ is a k-fold cover of $\mathbf{T}^2$. It is not hard to compute the genus of such a curve and we find that $g = kN + 1$. The moduli space of such curves has $2kN + 2$ (real) dimensions which are the large directions. It is also possible to identify the small dimensions. It actually true that the $(4Nk)$-dimensional moduli space $\mathcal{M}_{N,k}$ of instantons on $\widehat{\mathbf{T}}^2 \times \hat{\widetilde{\mathbf{T}}}^2$ for any size $\widehat{\mathbf{T}}^2$ is fibered over the $(2Nk+2)$-dimensional moduli space of curves inside $\mathbf{T}^2 \times \hat{\widetilde{\mathbf{T}}}^2$. This is a special case of the *spectral curve* construction (see [42] for details) and the remaining data that needs to be given is a line-bundle over the curve Σ, which is the spectral bundle. We will explain in a later section why the dimension of this moduli space differs from the dimension of the Coulomb branch.

3.3 Low-dimensional SYM

The matrix models for $d > 3$ are ill-defined since the field theories are not asymptotically free. For d=4, the matrix model coupling is,

$$g_5^2 = \frac{1}{M_{pl}^4 R_{\|} R_1 R_2 R_3 R_4}, \tag{3.9}$$

and determines a length scale in the problem. The moduli space of $4 + 1$-dimensional Yang-Mills has a metric which is linear in the moduli [43]. We need to check whether at energies obeying (3.2), the dynamics is determined by the UV or IR behavior of the theory. We compare,

$$\frac{g_5^2}{\Sigma_i} = (\hat{g}_3)^2 R_i \to 0,$$

which implies that the effective coupling is weak at the scale set by $\widehat{\mathbf{T}}^4$. By similar reasoning to the previous cases, we can then reduce to quantum mechanics on the moduli space of the $4+1$-dimensional theory. After reducing to quantum mechanics, it is natural to rescale the moduli absorbing the volume of $\widehat{\mathbf{T}}^4$ and the coupling constant. In terms of the rescaled variables $\tilde{\phi}$, the kinetic term is of the form:

$$(1 + R_{||}(\hat{g}_3)^4 \tilde{\phi})(\frac{d\tilde{\phi}}{dt})^2. \tag{3.10}$$

This quantum mechanics describes the physics below energies of order $1/R_i$ but includes energies around $\hat{g}_3^2$.

At first sight, in the conformal limit, it seems that the second term in (3.10) dominates. However, on closer inspection, we see that the moduli space has singularities at finite distance. For example, in the case of $SU(2)_1 \times SU(2)_2$ with hypermultiplets, the two effective gauge couplings h_1 and h_2 depend on the two scalars, ϕ_1 and ϕ_2, in the vector multiplets in the following way:

$$\frac{1}{h_1^2} = \frac{1}{g_5^2} + |\phi_1| - |\phi_2|, \qquad \frac{1}{h_2^2} = \frac{1}{g_5^2} - |\phi_1| + |\phi_2|.$$

It seems that for the metric to remain positive definite, the combination $|\phi_1| - |\phi_2|$ is bounded by $1/g_5^2$. If true, then the low-energy dynamics is described by the combination $|\phi_1| + |\phi_2|$ with a flat metric. This is quite puzzling and we will return to this point in section four after we discuss the DLCQ Wilson line.

For the case of $d = 5$, we need to understand the low-energy behavior of NS five-branes at the A_{k-1} singularity [29] which is quite difficult. Instead, we will discuss the matrix model for 1+1D SYM from a different route in the following section.

4 Exploring features of the matrix description

Now that we have arrived at the matrix model for 3+1D $\mathcal{N} = 4$ SYM from various viewpoints, we are ready to explore how the known properties of SYM are visible in this model. The main advantage of this DLCQ model is that S-duality is manifest. It is therefore interesting to see how perturbative features of SYM emerge.

4.1 Summary of the model

We have argued that the $p_{||} = N$ DLCQ sector of 3+1D $SU(k)$ SYM with both transverse directions compactified on $\mathbf{T}^2$ is given by quantum mechanics on a certain curved manifold $\widetilde{M}_{N,k}$ of (real) dimension $2Nk$. The space $\widetilde{M}_{N,k}$ is described as follows: start with the product space $\mathbf{T}^2 \times \tilde{\mathbf{T}}^2$ where $\tilde{\mathbf{T}}^2$ has complex structure τ. In the Coulomb branch approach, we only saw the limit $\mathbf{T}^2 \rightarrow \mathbb{R}^2$. Let $\widetilde{M}'_{N,k}$ be the moduli space of holomorphic Riemann surfaces Σ in $\mathbf{T}^2 \times \tilde{\mathbf{T}}^2$ which intersects k times any $\{p\} \times \tilde{\mathbf{T}}^2$ with p a generic point in the transverse space, and N times any $\mathbf{T}^2 \times \{p'\}$ again for generic $\{p'\}$. This space $\widetilde{M}'_{N,k}$ has dimension $2Nk + 2$ and admits a torus action $(U(1) \times U(1))$ which translates the curve Σ along $\tilde{\mathbf{T}}^2$. To obtain $\widetilde{M}_{N,k}$, we quotient $\widetilde{M}'_{N,k}$ by this torus action. Let

$$q^1 \dots q^{g-1}, \qquad g \equiv kN + 1$$

be local complex coordinates on the moduli space $\widetilde{M}_{N,k}$ of such surfaces. Locally, near a point $z \in \mathbf{T}^2$, the curve can be described by k functions from the neighborhood of z into $\widetilde{\mathbf{T}}^2$:

$$w_r(z,q) : \mathbf{T}^2 \longrightarrow \widetilde{\mathbf{T}}^2 \qquad r = 1 \dots k$$

and q is a shorthand for $q^1 \dots q^{g-1}$. These functions define a Riemann surface $\Sigma \subset \mathbf{T}^2 \times \widetilde{\mathbf{T}}^2$ of genus $g = kN+1$. The metric for the 0+1D Lagrangian $\left(\int \frac{1}{2} g_{i\bar{j}} \dot{q}^i \dot{\overline{q}}^{\bar{j}} + \dots\right)$ can be written as

$$g_{i\bar{j}} = R_{\|}{}^{-1} \sum_{r=1}^{k} \int d^2 z \left(\frac{\partial w_r}{\partial q^i} \cdot \frac{\partial \overline{w}_r}{\partial \overline{q}^{\bar{j}}} \right). \tag{4.1}$$

We have normalized the area of $\widetilde{\mathbf{T}}^2$ to be one. The local 1-forms,

$$\frac{\partial w_r}{\partial q^i} dz, \qquad i = 1 \dots g-1,$$

can be patched to a global holomorphic 1-form μ_i on Σ. So we can write,

$$g_{i\bar{j}} = R_{\|}{}^{-1} \int \mu_i \wedge \overline{\mu}_{\bar{j}}.$$

Under a complex supersymmetric variation δ, we see that

$$\delta q^i = \epsilon_a \psi^{ia},$$

The holomorphic spinors ψ^{ia} have spin-$\frac{1}{2}$ under the local rotation group of the transverse space $\mathbf{T}^2$ and the index $a = 1 \dots 4$ is a spinor of the $SO(6)$ R-symmetry. As usual, there are quadratic and quartic fermion terms which provide the usual supersymmetric completion of the sigma model.

4.2 DLCQ Wilson line

When we take the limit $\mathbf{T}^2 \to \mathbb{R}^2$, the vacua of the system should be characterized by the value of the $U(k)$ Wilson line along the light-like direction. In order to see this variable in the matrix description, we note that as $\mathbf{T}^2 \to \mathbb{R}^2$ the Riemann surfaces $\Sigma \subset \mathbb{R}^2 \times \widetilde{\mathbf{T}}^2$ are no longer compact. We now have to specify the boundary condition at the boundary of $\mathbb{R}^2$. These boundary conditions are k limiting points on $\widetilde{\mathbf{T}}^2$. The projection of these points on one axis of $\widetilde{\mathbf{T}}^2$ is in one-to-one correspondence with the conjugacy class of the $U(k)$ Wilson line which is specified by a point in $(\mathbf{S}^1)^k/S_k$. The projection on the other axis corresponds to "magnetic" Wilson lines or, in other words, the VEV of the dual photon since the system is effectively 2+1D.

In this limit, we then freeze $2k-2$ (real) moduli. The original moduli space $\widetilde{M}_{N,k}$ had dimension $2Nk$. The final moduli space then has $2k(N-1)+2$ parameters while on the other hand, the Coulomb branch picture led to a moduli space with dimension $2k(N-1)$. The discrepancy is accounted for by noting that the two parameters correspond to translation of the curve along $\mathbb{R}^2$. In the Coulomb picture, this center of mass motion corresponds to the diagonal $U(1)$ factor from (2.1) which had no charged matter, and therefore remains a modulus in 3+1D.

4.3 Electric and magnetic fluxes

The curves Σ becomes singular on a subspace $H_{N,k}$. In the Coulomb branch picture, the singular locus corresponds to the points where the Higgs and Coulomb branches meet. What boundary conditions should we impose on the wavefunctions at the singular locus $H_{N,k}$? We will provide a partial answer to this question.

We want to argue that $\pi_1(\widetilde{M}_{k,N} - H_{N,k})$ is non-trivial, and contains $\mathbb{Z}_k \times \mathbb{Z}_k$. To see this, note that $\widetilde{M}'_{k,N}$ has paths connecting two points which are identified under the torus action which gives $\widetilde{M}_{k,N}$. Intuitively, this path is given by smoothly varying the moduli and couplings of the $U(N)_i$ factor into those of the $U(N)_{i+1}$ factor cyclically around the chain (2.1). This means that we can consider sectors where the wavefunctions pick up a phase in $\mathbb{Z}_k \times \mathbb{Z}_k$ when taken around this loop in moduli space. To what can these boundary conditions correspond?

3+1D $SU(k)$ SYM compactified on $\mathbf{T}^3$ has different sectors specified by the discrete $\mathbb{Z}_k$ electric and magnetic fluxes along the sides of $\mathbf{T}^3$ [40]. If the gauge group were $U(k)$ instead of $SU(k)$, there would also be fluxes for the center $U(1)$ subgroup. The flux for the $SU(k) \subset U(k)$ is correlated with the $U(1)$ flux and the sector with $p \in \mathbb{Z}_k$ units of $SU(k)$ flux, either electric or magnetic, has to have a $U(1)$ flux in $\mathbb{Z} + \frac{p}{k}$ in the same direction. The $U(k)$ fluxes are characterized by the $U(1)$ fluxes in $\mathbb{Z}$ except that the quantization is in units of $\frac{1}{k}$. In the DLCQ, we put $U(k)$ SYM on $\mathbf{T}^2 \times \mathbf{S}^1$ where $\mathbf{S}^1$ is light-like.

Let $\mathbf{T}^2$ be with sides of sizes R_1, R_2 and $\mathbf{S}^1$ is, as usual, of size $R_{\|}$. In principle, we can have electric and magnetic fluxes either in the $\mathbf{S}^1$ direction, or in the $\mathbf{T}^2$ direction, or both. In the DLCQ, we can only observe the fluxes in the direction of $\mathbf{S}^1$. Fluxes in the transverse directions have an energy proportional to the inverse of the invariant length of $\mathbf{S}^1$, which becomes infinite as $\mathbf{S}^1$ becomes light-like. Another way to see this is to note that electric fluxes in a certain direction are the canonical dual variables to Wilson lines in this direction. In the DLCQ, this Wilson line is like a zero mode of a field (i.e. with $p_{\|} = 0$) and we cannot quantize the zero modes in the DLCQ. We claim that the twisted sectors where the wavefunction picks up a phase $e^{2\pi i (n_1/k)}$ for one generator of $\mathbb{Z}_k$ and $e^{2\pi i (n_2/k)}$ for the second generator correspond to electric and magnetic fluxes along the DLCQ direction.

Let us consider what changes if the gauge group were $U(k)$ rather than $SU(k)$. We would not be able to see the propagating modes for the $U(1)$ factor in this approach, but since the propagating $U(1)$ modes decouple from the $SU(k)$ theory, we do not lose much. However, the change to $U(k)$ does change the global structure by shifting the energy of the twisted sectors. The sectors with different fluxes are no longer degenerate, but have energies shifted by an amount proportional to,

$$\frac{1}{k}\tau_2 \frac{|n_1 + n_2\tau|^2}{R_1 R_2}. \tag{4.2}$$

What changes in the matrix model construction? From route three described in section two, we can see that this requires replacing $\widetilde{M}_{N,k}$ by $\widetilde{M}'_{N,k}$. The change amounts to introducing two extra collective coordinates which correspond to translating the curves Σ along $\widetilde{\mathbf{T}}^2$. Quantizing these extra collective coordinates gives the energy levels in (4.2).

4.4 W-bosons in the perturbative limit

How do we see the W-bosons in the matrix model? Since we are at the origin of the moduli space, the theory is conformal and strictly speaking there are no asymptotic states. Therefore W^0 and $W^\pm$ do not make sense as particles.

Since we have a generic Wilson line in the DLCQ, the charged W-particles become effectively massive and we can try to look for them. For simplicity let us work with $SU(2)$. A DLCQ Wilson line can be specified by $0 < \alpha < 1$. In the presence of the Wilson line, charged particles do not have integral $p_{||}$ anymore (see also [44]). The W^+ particles can have

$$p_{||} = \alpha, 1+\alpha, 2+\alpha, \dots$$

while the W^- particles can have

$$p_{||} = 1-\alpha, 2-\alpha, 3-\alpha, \dots.$$

In the presence of the Wilson line, the sector with $p_{||} = 2$, for example, can have 4 particles: two W^+ particles with $p_{||} = \alpha$ and two W^- particles with $p_{||} = 1-\alpha$.

It is easier to search for the W particles in the perturbative limit. In the perturbative limit, the auxiliary torus $\widetilde{\mathbf{T}}^2$ becomes very elongated. To see what the curves Σ look like, it is most useful to recall that these are the Seiberg-Witten curves of the $\mathcal{N}=2$ $SU(N)_1 \times SU(N)_2$ gauge theory with matter in the $(N, \overline{N})$ and $(\overline{N}, N)$ and coupling constants,

$$\tau_1 = \tau\alpha, \qquad \tau_2 = \tau(1-\alpha), \tag{4.3}$$

where for simplicity we have set the θ angles to zero. For the same reason, let us set $N = 2$. In the perturbative limit, the moduli space is parameterized by the VEVs of the two vector multiplets $\pm z_1$ and $\pm z_2$, respectively. The leading term in the metric over the moduli space is the classical term which implies the following kinetic energy for the matrix model:

$$\alpha|\dot{z}_1|^2 + (1-\alpha)|\dot{z}_2|^2.$$

We interpret this as two W^+-bosons with $p_{||} = \alpha$ at transverse positions z_1 and $-z_1$ and two W^--bosons with $p_{||} = 1-\alpha$ at transverse positions $\pm z_2$.

4.5 Coulomb interaction of W-bosons

We would also like to reproduce the interaction between W-bosons. It is well-known that the force between two W^+-bosons is velocity dependent. The static force generated by the exchange of photons and Higgs particles cancel. However, the static force between a W^+ and a W^--boson is not cancelled and it is interesting to see how this force is reproduced in the matrix model.

When a positively charged particle and a negatively charged particle are both parallel and moving toward each other at the speed of light, the electric and magnetic interactions exactly cancel. Thus, in the DLCQ the electro-magnetic potential energy between two particles with charges Q_1 and Q_2, at transverse positions z_1 and z_2, is given by:

$$V \sim Q_1 Q_2 |\dot{z}_1 - \dot{z}_2|^2 \log |z_1 - z_2|.$$

Note that $\log |z_1 - z_2|$ is the solution to Poisson's equation in the transverse two dimensions and that the expression $|\dot{z}_1 - \dot{z}_2|^2$ is Galilean invariant. After adding the Higgs exchange contribution to this expression, we find that the potential energy for a pair of W^+-bosons, or a W^+ and a W^- are both proportional to:

$$|\dot{z}_1 - \dot{z}_2|^2 \log |z_1 - z_2|.$$

In the matrix model, these expressions should be obtained as corrections to the metric on the moduli space. Viewed as the moduli space of $SU(N)_1 \times SU(N)_2$ gauge theory where $N = 2$, the first-order correction is a 1-loop effect which indeed behaves like,

$$|\dot{z}_1 - \dot{z}_2|^2 \log |z_1 - z_2|, \tag{4.4}$$

where z_1 and z_2 are VEVs for the scalar partners of the gauge fields; we are in the perturbative limit here. The difference between the energy of two like particles and the energy of opposite particles is accounted for by noting that for two W^+ bosons, both z_1 and z_2 correspond to scalar VEVs of the same $SU(N)_1$ factor and the term (4.4) comes from a loop containing the vector multiplet and bifundamentals. For a W^+ and a W^-, z_1 and z_2 belong to different $SU(N)$ factors and (4.4) is the effect of a hypermultiplet.

4.6 Enhanced R-symmetries

So far we have dealt with perturbative effects. We now turn to a non-perturbative effect which appears at strong coupling. This is the enhanced $Spin(8)_R$ symmetry in the IR limit of 2+1D SYM with $\mathcal{N} = 8$.

Let us recall some facts about R-symmetry in theories with 16 supersymmetries (see [5] for more details). R-symmetry is a global symmetry which acts on the supersymmetry generators. In theories with $\mathcal{N} = 4$ in 3+1D the maximal R-symmetry possible is $Spin(6) \equiv SU(4)$ since we have 4 complex generators. For 3+1D gauge theories this is indeed a symmetry and the 6 scalars transform in the **6** of $Spin(6)$. On the other hand, if we take the 5+1D $(2,0)$ theory and compactify on $\mathbf{T}^2$, we obtain a 3+1D theory with 16 supersymmetries in the uncompactified dimensions. This theory only has $Spin(5)$ R-symmetry. Its moduli space is not $\mathbb{R}^6$ but $\mathbb{R}^5 \times \mathbf{S}^1$ and there is no $Spin(6)$ symmetry. In the low-energy limit, however, the full $Spin(6)$ is restored. Similarly, 2+1D SYM with $\mathcal{N} = 8$ has a manifest $Spin(7)$ R-symmetry. In the IR limit, this theory flows to a non-trivial fixed point with an enhanced $Spin(8)$ R-symmetry. We could have also started with either the $(1,1)$ or $(2,0)$ 5+1D string theories compactified on $\mathbf{T}^3$. This theory generically has only a $Spin(4)$ R-symmetry but in the IR limit has an enhanced $Spin(8)$ R-symmetry. Similarly, the 5+1D theory on $\mathbf{T}^2$ has generically an $Spin(4)$ symmetry which is enhanced to $Spin(6)$ in the IR limit. The $Spin(6)$ symmetry of 3+1D SYM is, of course, a perturbative feature but the enhanced $Spin(8)$ of 2+1D SYM at low-energies is non-perturbative.

In the matrix model, we can see some of these phenomena to a greater or lesser extent. We have argued that the model for the non-critical string theory compactified on $\mathbf{T}^2$ is generically some 1+1D CFT at low-energies. It is obtained by compactifying a 5+1D theory on $\mathbf{T}^4$. Generically, the $Spin(4)$ rotation symmetry of the 5+1D theory in the directions of the $\mathbf{T}^4$ is broken. However, we have seen in section (2) that under certain conditions the matrix model for the low-energy limit of the 3+1D or 2+1D theory is given by the low-energy limit of a 3+1D theory compactified on $\mathbf{T}^3$, or a 4+1D theory compactified on $\mathbf{T}^4$, respectively. In the case of 2+1D theories with 16 supersymmetries, we obtained a 4+1D matrix model compactified on $\mathbf{T}^4$. In the low-energy limit the size of the $\mathbf{T}^4$ was very large compared to the scale of the theory and we could reduce to quantum mechanics on the moduli space. In the IR limit, the $Spin(4)$ rotation symmetry in the directions of $\mathbf{T}^4$ was restored even though the full spectrum (which was sensitive to the finite size of the $\mathbf{T}^4$) was not $Spin(4)$ symmetric. This $Spin(4)$ symmetry acts only on the fermionic variables of the quantum mechanics. The fermions also have quantum numbers under the $SU(2)$ R-symmetry of the 4+1D theory. This gives an obvious $Spin(4) \times Spin(3) \subset Spin(7)$ symmetry. Inspection of the Lagrangian reveals that the symmetry is the full $Spin(7)$. This is because every fermionic variable is in the

$$(\mathbf{2},\mathbf{1}) \times \mathbf{2} + (\mathbf{1},\mathbf{2}) \times \mathbf{2}$$

of $Spin(4) \times Spin(3)$ and thus comes with a multiplicity of 8. Let $\lambda^{\alpha i}$ be a fermionic variable with $\alpha = 1,2$ and $i = 1,2$ and let $\lambda^{\dot{\alpha} i}$ be the other variables. The kinetic energy terms are proportional to

$$\epsilon_{ij}\epsilon_{\alpha\beta}\lambda^{\alpha i}\partial_t\lambda^{\beta j} + \epsilon_{ij}\epsilon_{\dot{\alpha}\dot{\beta}}\lambda^{\dot{\alpha} i}\partial_t\lambda^{\dot{\beta} j}$$

which is $Spin(8)$ symmetric. The quartic terms on the other hand look like

$$D^m D^m, \qquad m = 1\ldots 3,$$

where

$$D^m = \epsilon_{\alpha\beta}\sigma^m_{ij}\lambda^{\alpha i}\lambda^{\beta j} - \epsilon_{\dot{\alpha}\dot{\beta}}\sigma^m_{ij}\lambda^{\dot{\alpha} i}\lambda^{\dot{\beta} j}.$$

A tedious calculation shows that $\lambda^{\alpha i}$ and $\lambda^{\dot{\alpha} i}$ can be combined into a spinor λ of $Spin(7)$ which enlarges $Spin(4) \times Spin(3)$ and

$$D^m D^m = (\lambda\Gamma^A\lambda)(\lambda\Gamma_A\lambda), \qquad A = 1\ldots 7,$$

where Γ^A are Dirac matrices of $Spin(7)$. Thus the R-symmetry of the quantum mechanics is indeed $Spin(7)$. It cannot be $Spin(8)$ since $Spin(8)$ does not have an anti-symmetric quartic invariant as can be seen by triality.

How does $Spin(8)$ appear in the IR limit of the 2+1D SYM? We found in section *3.3* that the IR limit seemed to be described by a flat metric. In this case, the $Spin(8)$ violating curvature term vanishes and we trivially recover a $Spin(8)$, but not the $Spin(8)$ of the spacetime SYM theory. A probable resolution to this puzzle is that we are describing a model with a DLCQ Wilson line. The Wilson line distinguishes the dual photon from the other 7 scalars, and so breaks $Spin(8)$. With the Wilson line, the theory will now flow to a free theory in the IR rather than the interacting fixed point. To recover a model with $Spin(8)$, we need to describe the case without a DLCQ Wilson line. A possible way to turn off the Wilson line is to tune α in (4.3) to zero, which takes one of the bare couplings to infinity. We will not explore this possibility further here.

Similarly, the 3+1D theory had a matrix model which was a 3+1D theory compactified on $\mathbf{T}^3$. In the IR limit, the rotation group $Spin(3) \equiv SU(2)$ of $\mathbf{T}^3$ was restored. The fermionic variables in the quantum mechanics now have a spinor index under this $SU(2)$ and another spinor index under the $SU(2)$ R-symmetry of the 3+1D matrix quiver theory. It should now be the case that there is a full $SU(4)$ symmetry mixing all four indices.

4.7 1+1D SYM

The lowest dimension where a DLCQ description exists is 1+1D. Compactifying the $U(k)$ 5+1D $(2,0)$ field theory on $\mathbf{T}^4$ gives a 1+1D theory which flows in the IR to the orbifold $(\mathbb{R}^5 \times \mathbf{T}^3)^k/S_k$. In a limit, we recover the orbifold $(\mathbb{R}^8)^k/S_k$ which describes the IR limit of 1+1D SYM.

The spacetime SYM theory has a DLCQ Wilson line as before, which we conjecture flows in the IR to a particular S_k element in the longitudinal direction. To describe the DLCQ sector for the free orbifold $(\mathbb{R}^8)^k/S_k$ with $p_{||} = N$, we need to specify this S_k element. This discrete Wilson line is specified by the number of cycles each of length,

$$l_1 \dots l_k, \qquad \sum j l_j = k.$$

In the 1+1D IR limit, all the particles are either left-moving or right-moving at the speed of light. We will see that the matrix description in terms of the moduli space $\mathcal{M}_{k,N}$ of $U(k)$ instantons on $\mathbf{T}^4$ can reproduce the multiplicity of states of the, for example, right-moving modes of the orbifold $(\mathbb{R}^8)^k/S_k$ with the discrete S_k Wilson line σ given by a single cycle of length k. To see this, we note that since there are no transverse directions in 1+1D all states in the IR limit have DLCQ energy of precisely N. This means that to count the number of states in the 1+1D IR limit, we need to count the number of vacua in the quantum mechanics on $\mathcal{M}_{k,N}$. The result of this computation can be predicted by U-duality and there is some supporting evidence for the prediction [45]. We can summarize the result for the cohomology of $\mathbf{T}^4$ which contains 8 even-dimensional elements and 8 odd-dimensional elements in the following way: we associate a bosonic field ϕ^a_{-n} to the even-dimensional elements ($a = 1 \dots 8$) and a fermionic field ψ^b_{-n} to the odd-dimensional elements ($b = 1 \dots 8$) where $n = 1, 2, \dots$ is the oscillator number. The cohomology of the resolution of $\mathcal{M}_{k,N}$ is conjectured to be given by the states of the Fock space generated by ϕ^a_{-n}, ψ^b_{-n} at level Nk. This agrees with the degeneracy of right-moving states of $(\mathbb{R}^8)^k/S_k$ with the special Wilson line σ which is generated by 8 oscillators of fractional $p_{\|} = \frac{j}{k}$ $(j = 1, 2, \ldots)$ which sum up to N.

5 Concluding comments

Among the cases we have described is 3+1D SYM compactified on $\mathbf{T}^2$. This is effectively a 1+1D theory and as such has no moduli space. In the decompactification limit, when $\mathbf{T}^2 \to \mathbb{R}^{3,1}$, we also have to specify the point in the moduli space. There are two qualitative choices: the first leads to a free theory in the IR, while the second leads to an interacting theory. In the first case, we can have particle states and so with a matrix description, we can try to study the scattering of charged particles. We would hope to obtain a non-perturbative prescription for computing S-matrix elements involving photons, W-bosons and dyons. However, as we have seen, it is easiest to find a matrix model for the conformal theory.

How do we introduce a VEV for the decompactified limit of SYM? In the DLCQ, the VEV should be an external parameter. We can first consider the quantum mechanics describing the $(2,0)$ theory [27], which contains a $U(N)$ vector multiplet, an adjoint hypermultiplet and k fundamental hypermultiplets. Separating the longitudinal five-branes corresponds to giving bare masses m_a, $a = 1, \ldots, k$ to the fundamental hypermultiplets. On taking the quantum mechanical coupling to infinity, the quartic potential terms still dominate so we are constrained to the moduli space of instantons, $\mathcal{M}_{N,k}$. The mass perturbations then descend to potentials on the moduli space of instantons. However, for the 3+1D theories the story seems to be more complicated. From the Coulomb branch perspective, we need to resolve the A_{k-1} singularity and this leads to problems which seem related to the issues raised in [46].

We can also try to construct similar models for theories with less supersymmetry. There are several possible approaches: one way SYM theories with $\mathcal{N} = 2$ in 3+1D have been realized is in terms of the low-energy degrees of freedom of five-branes wrapped on a Riemann surface $\Sigma \subset \mathbb{R}^3 \times \mathbf{S}^1$ in the limit $\mathbf{S}^1 \to 0$ [47, 37]. We can ask if there is a matrix model description for this construction. Let us view this as a limit of a longitudinal five-brane wrapped on $\mathbf{T}^2 \times \Sigma \subset \mathbf{T}^2 \times \mathbf{T}^4$. This makes the five-brane finite in the transverse directions. If we understood more fully the matrix model for M theory on $\mathbf{T}^6$, we could have identified this configuration as the sector with the appropriate quantum number specifying wrapped longitudinal five-branes. Nevertheless, we can assume that at low-energies, the matrix model looks like 6+1D SYM. Since we need the limit of $\mathbf{T}^6 \to 0$, the matrix model will be formulated on the dual, large $\widehat{\mathbf{T}}^6$. The relevant quantum numbers for longitudinal five-branes are encoded in the Pontryagin class $\mathrm{Tr} F \wedge F$. Therefore, we are looking for low-energy solutions of 6+1D SYM with a given Pontryagin class, a point which has also been discussed in [48].

To get the limit of 3+1D SYM, we have to take $\mathbf{T}^6 = \mathbf{T}^2 \times \widetilde{\mathbf{T}}^4$ where $\mathbf{T}^2$ is the transverse space and $\widetilde{\mathbf{T}}^4$ is auxiliary and very small. The matrix model is formulated on the dual $\widehat{\mathbf{T}}^2 \times \widehat{\widehat{\mathbf{T}}}^4$ and as in section (3), we may reduce to the moduli space of holomorphic maps from $\mathbf{T}^2$ to a divisor of $\widehat{\widehat{\mathbf{T}}}^4$. This means that for each point z of transverse space $\mathbf{T}^2$ there is Riemann surface $\Sigma_z \subset \widehat{\widehat{\mathbf{T}}}^4$ which varies holomorphically with $\mathbf{T}^2$. When the transverse space is decompactified to $\mathbb{R}^2$, the boundary conditions on the Riemann surface are such that Σ_z has to go over to the Seiberg-Witten curve of the theory as $z \to \infty$. On top of that we also need to specify a point on the Jacobian of each Riemann surface Σ_z, i.e. g points on Σ_z where g is the genus of the surface. Alternatively, g is also the number of $U(1)$-factors at a generic point in the low-energy theory. The boundary conditions on these holomorphically varying g points are such that as $z \to \infty$, they become electric and magnetic Wilson lines of the $U(1)^g$ low-energy photons along the longitudinal direction. Note that such Wilson lines are naturally identified with a point on the Jacobian. There are a number of problems with this possibility so the above comments should be taken as speculative. In a similar spirit, perhaps matrix models for $\mathcal{N} = 1$ theories might be constructed starting from the five-brane constructions of [49].

There is a second approach for obtaining matrix formulations of finite 3+1D $\mathcal{N} = 2$ theories. We can start with the theories described in [17] compactified on $\mathbf{T}^2$. Matrix models for these cases can be obtained by studying a system of 4-branes and 0-branes on *ALE* singularities. These models will have $(0, 4)$ supersymmetry, and the Coulomb branches of these models should again describe the conformal points of the SYM theories.

It is interesting to note that we have described SYM theories in terms of quantum mechanics. While describing the low-energy dynamics of SYM on a lattice has proven difficult because of chiral fermions, a numerical analysis of the quantum mechanics should be considerably simpler. After all, fermions in the quantum mechanics are simply large matrices. Lastly, it seems possible that some insight into the large N limit for 3+1D SYM can be obtained by studying how the curves that appeared in that matrix formulation behave in the large N limit. This might well be the case for the 't Hooft scaling limit, rather than the large N limit need for matrix theory.

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